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Partition-complete paracompact k-spaces are preserved by closed maps

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Abstract

The theorem in the title is proved, together with some related results. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The following theorem was proved by Vaĭnšteĭn in [19]. (See also Engelking [5, 4.5.13(e)].)

Theorem 1.1 (Vaĭnšteĭn). *If $f : X \to Y$ is a closed map [2] from a completely metrizable space X onto a metrizable space Y, then Y is also completely metrizable.*

The purpose of this note is to study the preservation by closed maps of two generalizations of complete metrizability, both of which coincide with complete metrizability in metrizable spaces: partition-completeness and, primarily for comparison, sieve-completeness. For definitions and references, see Section 2. Here we note that

Čech-complete → sieve-complete → partition-complete,

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[2] All *maps* in this paper are continuous.

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that the first arrow is reversible in paracompact spaces, and that all three classes are preserved by closed subsets. Moreover, scattered spaces are partition-complete, regular sieve-complete spaces are k-spaces, and regular partition-complete spaces are Baire spaces.

Part (a) of the following theorem generalizes Theorem 1.1.

Theorem 1.2. *Let $f : X \to Y$ be a closed map from a completely metrizable space X onto a space Y. Then*:

(a) *Y is partition-complete* [3].

(b) *Y is sieve-complete if and only if* Bdry $f^{-1}(y)$ *is compact for every $y \in Y$* (*in which case Y is completely metrizable*).

According to Theorem 1.2, closed maps treat sieve-completeness and partition-completeness quite differently, even when the domain is metrizable [4]. By contrast, these properties are treated identically—both being preserved without any restrictions—by open maps and perfect maps [10,18], and, more generally, by tri-quotient maps [10,14,15].

The following two theorems show how the two parts of Theorem 1.2 can be generalized.

Theorem 1.3. *Let $f : X \to Y$ be a closed map from a partition-complete, paracompact k-space X onto a space Y. Then Y is also partition-complete.*

Remark. Since paracompact spaces and k-spaces are both preserved by closed maps, Theorem 1.3 implies the result stated in the title of this paper.

Theorem 1.4. *Let $f : X \to Y$ be a closed map from a sieve-complete, paracompact space X onto a space Y. Then Y is sieve-complete if and only if* Bdry $f^{-1}(y)$ *is compact for every $y \in Y$.*

An example of Gruenhage and Watson in [7] implies that the k-space assumption in Theorem 1.3 cannot be omitted; see Example 7.1 below. An example of van Douwen in [3] implies that, assuming $\mathfrak{b} = \mathfrak{c}$ [5], the paracompactness assumption in Theorem 1.3 also cannot be omitted; see Example 7.2. In Theorem 1.4, finally, the paracompactness assumption can be omitted for "if" but not, by [9, Example 3.1], for "only if".

Section 2 is devoted to basic definitions. Some new characterizations of partition-complete spaces are obtained in Proposition 3.4, and these are used in Theorem 4.2 to show that a closed map $f : X \to Y$ with regular domain X preserves partition-completeness if $f|A$ is inductively irreducible for every closed $A \subset X$. In Section 5, that result is combined with theorems of Lašnev [8] and Gruenhage [7] to prove Theorems 1.2(a) and 1.3. It is also shown (see Theorem 5.4) that, for a closed map with paracompact domain, the irreducibility condition in Theorem 4.2 is not only sufficient but also necessary for the

[3] In a different direction, Van Doren showed in [20] that Y has a dense, completely metrizable subspace.

[4] Consider, for example, the closed map $f : R \to R/Z$ which identifies the integers Z in the reals R. By Theorem 1.2, R/Z is partition-complete but not sieve-complete.

[5] $\mathfrak{b} = \mathfrak{c}$ is a set-theoretic condition implied by the continuum hypothesis.

preservation of partition-completeness. Section 6 proves Theorems 1.2(b) and 1.4, and Section 7 is devoted to examples.

2. Basic definitions

A decreasing sequence (U_n) of subsets of a space X is *complete* if, whenever $\mathcal{F}$ is a filter base on X such that $F \cap U_n \neq \emptyset$ for all $F \in \mathcal{F}$ and all n, then $\mathcal{F}$ clusters at some $x \in X$. (For later application, we note that every complete sequence (U_n) is *countably complete* in the sense that, if $x_n \in U_n$ for all n, then the sequence (x_n) has a cluster point $x \in X$.)

A *sieve* on a space X is a sequence of indexed covers $\{U_\alpha\colon\ \alpha \in A_n\}$ $(n \geqslant 0)$ on X, together with functions

$$\pi_n : A_{n+1} \to A_n,$$

such that $U_\alpha = X$ for $\alpha \in A_0$ and

$$U_\alpha = \bigcup \left\{U_\beta\colon\ \beta \in \pi_n^{-1}(\alpha)\right\} \quad \text{for all } \alpha \in A_n \text{ and all } n.$$

Such a sieve is called *complete* if, whenever $\alpha_n \in A_n$ with $\pi_n(\alpha_{n+1}) = \alpha_n$ for all n, then the sequence (U_{α_n}) is complete.

A sieve $(\{U_\alpha\colon\ \alpha \in A_n\}, \pi_n)$ on X is *open* if every U_α is open in X. A space X is *sieve-complete* [10] (= monotone Čech-complete [1]) if it has a complete, open sieve.

A cover $\mathcal{U}$ of a space X is *exhaustive* [11] if every nonempty $S \subset X$ has a nonempty, relatively open subset of the form $U \cap S$ with $U \in \mathcal{U}$. (It is clearly sufficient if this condition is satisfied for every *closed*, nonempty $S \subset X$.) A sieve $(\{U_\alpha\colon\ \alpha \in A_n\}, \pi_n)$ on X is *exhaustive* if $\{U_\beta\colon\ \beta \in \pi_n^{-1}(\alpha)\}$ is an exhaustive cover of U_α for all $\alpha \in A_n$ and all n. A space X is *partition-complete* if it has a complete, exhaustive sieve [11,18].

Since open covers and sieves are clearly exhaustive, every sieve-complete space is partition-complete; the converse is generally false (see footnote 4). That Čech-complete spaces are sieve-complete, and conversely in paracompact spaces, is proved in [1,10]. That partition-completeness and complete metrizability coincide in metrizable spaces is proved in [11,12].

3. Some characterizations of partition-complete spaces

The principal purpose of this section is to prove Proposition 3.4.

Call a cover $\mathcal{U}$ of a space Y *pseudo-exhaustive* if for every nonempty $S \subset X$ there is a $U \in \mathcal{U}$ such that $\mathrm{Int}_S(U \cap S) \neq \emptyset$. (It clearly suffices if this condition is satisfied for every *closed* $S \subset X$.) Call a sieve $(\{U_\alpha\colon\ \alpha \in A_n\}, \pi_n)$ on X *pseudo-exhaustive* if $\{U_\beta\colon\ \beta \in \pi_n^{-1}(\alpha)\}$ is a pseudo-exhaustive cover of U_α for all $\alpha \in A_n$ and all n. Clearly every exhaustive cover (respectively sieve) is pseudo-exhaustive.

Call a cover $\mathcal{U}$ of X *hereditary* if $V \subset U \in \mathcal{U}$ implies $V \in \mathcal{U}$. Call a sieve $(\{U_\alpha\colon\ \alpha \in A_n\}, \pi_n)$ on X *hereditary* if $\{U_\beta\colon\ \beta \in \pi_n^{-1}(\alpha)\}$ is a hereditary cover of U_α for all $\alpha \in A_n$ and all n. Note that a hereditary cover (respectively sieve) is exhaustive if and only if it is pseudo-exhaustive.

Lemma 3.1. *Let* $(\{U_\alpha\colon \alpha \in A_n\}, \pi_n)$ *be a sieve on* X. *Then there exists a hereditary sieve* $(\{V_{\alpha,\lambda}\colon (\alpha,\lambda) \in A_n \times \Lambda_n\}, \pi_n \times \varphi_n)$ *on* X (*where* $\varphi_n : \Lambda_{n+1} \to \Lambda_n$) *such that, for all* $(\alpha,\lambda) \in A_n \times \Lambda_n$:

(a) $V_{\alpha,\lambda} \subset U_\alpha$.

(b) *If* $\beta \in \pi_n^{-1}(\alpha)$, *then* $U_\beta \cap V_{\alpha,\lambda} = V_{\beta,\mu}$ *for some* $\mu \in \varphi_n^{-1}(\lambda)$.

Proof. Let $\Lambda_0 = \{\lambda_0\}$, and let $V_{\alpha,\lambda_0} = X$ for all $a \in A_0$. Suppose we have everything up to n. Pick a set Γ with $\operatorname{card}\Gamma = \exp(\operatorname{card} X)$, let $\Lambda_{n+1} = \Lambda_n \times \Gamma$, and define $\varphi_n : \Lambda_{n+1} \to \Lambda_n$ by $\varphi_n(\lambda,\gamma) = \lambda$. Then, for each $(\alpha,\lambda) \in A_n \times \Lambda_n$ and $\beta \in \pi_n^{-1}(\alpha)$, let $\{V_{\beta,\mu}\colon \mu \in \varphi_n^{-1}(\lambda)\}$ be an indexing of $\{E\colon E \subset (U_\beta \cap V_{\alpha,\lambda})\}$. It is easy to check that this works. $\square$

Lemma 3.2. *Let* $\mathcal{U}$ *be a hereditary, exhaustive cover of a regular space* X. *Then* $\{A \in \mathcal{U}\colon A$ *closed in* $X\}$ *is a pseudo-exhaustive cover of* X.

Proof. We must show that, if $S \neq \emptyset$ is closed in X, then $\operatorname{Int}_S(W \cap S) \neq \emptyset$ for some $W \in \mathcal{U}$. Pick $U \in \mathcal{U}$ such that $U \cap S$ is a nonempty, relatively open subset of S. Since S is regular, there is a nonempty, relatively open V in S such that $\overline{V} \subset U$. But now $\overline{V} \in \mathcal{U}$ because $\mathcal{U}$ is hereditary, and $\operatorname{Int}_S(\overline{V} \cap S) = \operatorname{Int}_S \overline{V} \supset V \neq \emptyset$. $\square$

In the following result, a sieve $(\{U_\alpha\colon \alpha \in A_n\}, \pi_n)$ on X is called *closed* if U_α is closed in X for all $\alpha \in A_n$ and all n.

Lemma 3.3. *Let* $(\{U_\alpha\colon \alpha \in A_n\}, \pi_n)$ *be a* (*complete*) *hereditary, exhaustive sieve on a regular space* X. *Then there are* $A'_n \subset A_n$ *such that* $(\{U_\alpha\colon \alpha \in A'_n\}, \pi_n | A'_{n+1})$ *is a* (*complete*) *pseudo-exhaustive, closed sieve on* X.

Proof. We define the A'_n inductively by letting $A'_0 = A_0$ and defining

$$A'_{n+1} = \bigl\{\beta \in \pi_n^{-1}(A'_n)\colon U_\beta \text{ closed in } X\bigr\}.$$

It follows from Lemma 3.2 that these A'_n have the desired property. $\square$

Proposition 3.4. *The following are equivalent for a regular space* X.

(a) X *has a complete, hereditary, exhaustive sieve.*

(b) X *has a complete, exhaustive sieve* (*i.e.,* X *is partition-complete*).

(c) X *has a complete, pseudo-exhaustive sieve.*

(d) X *has a complete, closed, pseudo-exhaustive sieve.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (c) Clear.

(c) $\Rightarrow$ (a) Let $(\{U_\alpha\colon \alpha \in A_n\}, \pi_n)$ be a sieve on X satisfying (c). Then it is easy to check that the sieve $(\{V_{\alpha,\lambda}\colon (\alpha,\lambda) \in A_n \times \Lambda_n\}, \pi_n \times \varphi_n)$ in Lemma 3.1 satisfies (a).

(a) $\Rightarrow$ (d) This follows from Lemma 3.3. $\square$

Remark. In the above proof, regularity is needed only to prove (a) $\Rightarrow$ (d).

4. Irreducible maps and feebly open maps

We begin by recalling two definitions. A map $f : X \to Y$ onto Y is *irreducible* if $f(A) \neq Y$ for every closed $A \subset X$ with $A \neq X$. A map $f : X \to Y$ is *feebly open* [6,13] if $\operatorname{Int} f(U) \neq \emptyset$ for every nonempty open $U \subset X$ (equivalently: if $\operatorname{Int} f(U) \neq \emptyset$ whenever $\operatorname{Int} U \neq \emptyset$).

Lemma 4.1. *Every irreducible, closed, onto map $f : X \to Y$ is feebly open.*

Proof. If $U \neq \emptyset$ is open in X, then $f(U)$ contains the nonempty open set $Y \setminus f(X \setminus U)$. □

The following is the principal result of this section. We say that a map $f : X \to Y$ onto Y has a property P *inductively* if there is a closed $E \subset X$ such that $f(E) = Y$ and $f|E : E \to Y$ has property P.

Theorem 4.2. *Let $f : X \to Y$ be a map from a regular space X onto Y. Then* (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

(a) *f is closed, and $f|A : A \to f(A)$ is inductively irreducible for every closed $A \subset X$.*
(b) *$f|A : A \to f(A)$ is inductively feebly open for every closed $A \subset X$.*
(c) *If $A \subset X$ is closed, and if $\mathcal{U}$ is a pseudo-exhaustive cover of A, then $f(\mathcal{U})$ is a pseudo-exhaustive cover of $f(A)$.*
(d) *If $(\{U_\alpha \colon \alpha \in A_n\}, \pi_n)$ is a pseudo-exhaustive, closed sieve on X, then $(\{f(U_\alpha) \colon \alpha \in A_n\}, \pi_n)$ is a pseudo-exhaustive sieve on Y.*
(e) *If X is partition-complete, so is Y.*

Proof. (a) $\Rightarrow$ (b) Immediate from Lemma 4.1.

(b) $\Rightarrow$ (c) Assume (b). To prove (c), it clearly suffices to prove it for $A = X$. So let $\mathcal{U}$ be a pseudo-exhaustive cover of X, and let us show that $f(\mathcal{U})$ is a pseudo-exhaustive cover of Y. It will suffice to show that, if $S \neq \emptyset$ is closed in Y, then $\operatorname{Int}_S(f(U) \cap S) \neq \emptyset$ for some $U \in \mathcal{U}$.

Let $C = f^{-1}(S)$. Then C is closed in X and $f(C) = S$, so by (b) there is closed $E \subset C$ such that $f(E) = S$ and $f|E : E \to S$ is feebly open. Since $\mathcal{U}$ is a pseudo-exhaustive cover of X, there is a $U \in \mathcal{U}$ such that $\operatorname{Int}_E(U \cap E) \neq \emptyset$, and hence $\operatorname{Int}_S(f(U \cap E)) \neq \emptyset$. But $(f(U) \cap S) \supset f(U \cap E)$, so $\operatorname{Int}_S(f(U) \cap S) \neq \emptyset$.

(c) $\Rightarrow$ (d) Assume (c). Suppose that $(\{U_\alpha \colon \alpha \in A_n\}, \pi_n)$ is a pseudo-exhaustive, closed sieve on X, and let us show that $(\{f(U_\alpha) \colon \alpha \in A_n\}, \pi_n)$ is a pseudo-exhaustive sieve on Y. Since it is clearly a sieve on Y, we must show that, if $\alpha \in A_n$ for some n, then $\{f(U_\beta) \colon \beta \in \pi_n^{-1}(\alpha)\}$ is a pseudo-exhaustive cover of $f(U_\alpha)$. But that follows from (c), since U_α is closed in X and $\{U_\beta \colon \beta \in \pi_n^{-1}(\alpha)\}$ is a pseudo-exhaustive cover of U_α by hypothesis.

(d) $\Rightarrow$ (e) Assume (d). Suppose that X is partition-complete. Then X has a complete, closed, pseudo-exhaustive sieve $(\{U_\alpha \colon \alpha \in A_n\}, \pi_n)$ by Proposition 3.4, (b) $\Rightarrow$ (d). Hence $(\{f(U_\alpha) \colon \alpha \in A_n\}, \pi_n)$ is a pseudo-exhaustive sieve on Y by (d), and it is complete by [10, Lemma 4.1]. Thus Y is partition-complete by Proposition 3.4, (c) $\Rightarrow$ (b). □

Remark. In the above proof, regularity is needed only to prove (d) $\Rightarrow$ (e).

The implication (a) $\Rightarrow$ (e) in Theorem 4.2 will be applied in Section 5. It should be noted that this implication becomes false if (a) is changed to "f is closed and irreducible"; see Example 7.1.

We conclude this section with two results on irreducible maps, the second of which will be applied in Section 7.

Lemma 4.3. *Let $f : X \to Y$ be a map onto Y, and suppose that there is a dense $E \subset X$ such that $E = f^{-1}(f(E))$ and $f|E$ is one-to-one. Then f is irreducible.*

Proof. Suppose $C \subset X$ is closed and $f(C) = Y$. Then $C \supset E$ because $f(X) = Y$, so $C = X$ because E is dense and C is closed in X. $\square$

Proposition 4.4. *Suppose that $g : A \to B$ is a closed map onto B. Let $X = A \times S$, where $S = \{0\} \cup \{1/n\colon n = 1, 2, \ldots\}$, and identify A with $A \times \{0\} \subset X$. Then there exists a closed, irreducible map $f : X \to Y$ such that $f(A)$ is homeomorphic to B and closed in Y.*

Proof. Let $f : X \to Y$ be the quotient map obtained from the partition of X consisting of all sets $g^{-1}(y)$ with $y \in B$ and all $\{x\}$ with $x \in X \backslash A$. Then $f(A)$ is homeomorphic to B because g is a quotient map, $f(A)$ is closed in Y because $f^{-1}(f(A)) = A$, the map f is closed because the map g is closed, and f is irreducible by Lemma 4.3 with $E = X \backslash A$. $\square$

5. Proofs of Theorem 1.2(a) and its generalizations

In this section our principal theorems are proved by combining Theorem 4.2 with sufficient conditions for a closed map to be inductively irreducible. The simplest such condition, easily verified by a simple Zorn's lemma argument, is that the map be perfect. That yields no new information about partition-complete spaces, however, since, as indicated in the introduction, such spaces are already known to be preserved by perfect maps. For closed maps which are not assumed perfect, we have, first of all, the following result of Lašnev [8].

Theorem 5.1 (Lašnev). *Every closed map from a paracompact space X onto a Fréchet space Y is inductively irreducible.*

Proof of Theorem 1.2(a). Let $f : X \to Y$ be a closed map from a completely metrizable space X onto Y. Since metrizable spaces are Fréchet, and Fréchet spaces are preserved by closed maps, it follows from Theorem 5.1 that f satisfies condition 4.2(a). Since X, being completely metrizable, is partition-complete, it follows from Theorem 4.2 that Y is also partition-complete. $\square$

Lašnev's theorem was generalized in several directions by Chertanov [2] and Gruenhage [7]. The following result was obtained by Gruenhage in [7, Corollary 2.8].

Theorem 5.2 (Gruenhage). *Let $f : X \to Y$ be a closed map from a meta-Lindelöf* [6] *space X onto a regular space Y with the following property*:

(5.2.1) *Every dense-in-itself* [7]*, open $V \subset Y$ has a countable subset A with a cluster point in V.*

Then f is inductively irreducible.

The following lemma exhibits two kinds of spaces which satisfy (5.2.1). Part (a) will be applied in the proof of Theorem 5.4, and part (b) in the proof of Theorem 5.5.

Lemma 5.3. *Let Y be a regular space. Then condition* (5.2.1) *follows from either* (a) *or* (b) *below.*

(a) *Y is a quasi-k-space* [8].

(b) *Y is partition-complete.*

Proof. Let $V \subset Y$ be dense-in-itself and open, and let us show that some countable $A \subset V$ has an accumulation point in V.

Pick a nonempty, open $W \subset Y$ with $\overline{W} \subset V$.

(a) Choose $y \in W$, and let $S = W \setminus \{y\}$. Then S is not closed in Y, so $S \cap K$ is not relatively closed in K for some countably compact $K \subset Y$. Hence $S \cap K$ is infinite, so one can take A to be any countably infinite subset of $S \cap K$.

(b) Let $(\{U_\alpha\colon \alpha \in A_n\}, \pi_n)$ be a complete, exhaustive sieve on Y. By induction, choose $\alpha_n \in A_n$ such that $\pi_n(\alpha_{n+1}) = \alpha_n$ and $U_{\alpha_n} \cap W$ is open and nonempty for all n. Since W is dense-in-itself, the set $U_{\alpha_n} \cap W$ is infinite for all n, so we can choose distinct $y_n \in U_{\alpha_n} \cap W$ for all n. Since (U_{α_n}) is a complete—and hence countably complete—sequence of subsets of X, the infinite set $A = \{y_n\colon n = 1, 2, \ldots\}$ has a cluster point $y \in \overline{W} \subset V$. □

The following theorem implies Theorem 1.3.

Theorem 5.4. *Let $f : X \to Y$ be a closed map from a partition-complete, regular, meta-Lindelöf space X onto a regular quasi-k-space Y. Then Y is also partition-complete.*

Proof. By Theorem 4.2, it suffices to show that $f|A : A \to f(A)$ is inductively irreducible for every closed $A \subset X$. But since $f(A)$, being closed in Y, is also a quasi-k-space, that follows by applying Theorem 5.2 and Lemma 5.3(a) to the map $f|A : A \to f(A)$. □

[6] A space X is *meta-Lindelöf* if every open cover of X has a point-countable, open refinement.

[7] I.e., nonempty and without isolated points.

[8] A space Y is a (*quasi-*) *k-space* if a set $E \subset Y$ is closed in Y whenever $E \cap K$ is relatively closed in K for every (countably) compact $K \subset Y$.

Proof of Theorem 1.3. Since paracompact spaces are meta-Lindelöf, and since k-spaces (and quasi-k-spaces) are preserved by closed maps, Theorem 1.3 follows immediately from Theorem 5.4. □

Theorem 5.5. *Let $f : X \to Y$ be a closed map from a partition-complete, paracompact space X onto a space Y. Then the following are equivalent*:

(a) *$f|A : A \to f(A)$ is inductively irreducible for every closed $A \subset X$.*

(b) *Y is partition-complete.*

Proof. (a) $\Rightarrow$ (b) Immediate from Theorem 4.2.

(b) $\Rightarrow$ (a) Assume (b). Then every closed subset of Y is partition-complete, and thus satisfies (5.2.1) by Lemma 5.3(b). Hence (a) follows from Theorem 5.2 applied to $f|A : A \to f(A)$. □

The implication (a) $\Rightarrow$ (b) in Theorem 5.5 becomes false if (a) is weakened to "f is inductively irreducible"; see Example 7.1. We do, however, have the following analogue of Theorem 5.5, where a regular space is called *almost complete* (see [13, Section 2]) if it has a dense, partition-complete subspace. Unlike partition-complete spaces, almost complete spaces are not preserved by closed subsets, but they are preserved by irreducible, closed maps and, more generally, by feebly open maps [13, Proposition 6.3].

Proposition 5.6. *Let $f : X \to Y$ be a closed map from a partition-complete, paracompact space X onto a space Y. Then the following are equivalent*:

(a) *f is inductively irreducible*;

(b) *Y is almost complete*;

(c) *Y satisfies* (5.2.1).

Proof. (a) $\Rightarrow$ (b) Assume (a). Pick a closed $A \subset X$ such that $f(A) = Y$ and $f|A$ is irreducible. Then A is partition-complete because A is closed in X, hence A is almost complete, and thus Y is also almost complete.

(b) $\Rightarrow$ (c) Essentially the same proof as for Lemma 5.3(b).

(c) $\Rightarrow$ (a) By Theorem 5.2. □

Remark. Example 7.3 shows that "partition-complete" cannot be weakened to "almost complete" in the hypothesis of Proposition 5.6.

6. Proofs of Theorems 1.2(b) and 1.4

We obtain both of these theorems as parts of Theorem 6.2 below. First, a definition and a lemma.

Recall that a space Y is a *q-space* [9] if each $y \in Y$ has a countably complete (see Section 2) sequence of neighborhoods in Y. First-countable spaces and locally countably compact spaces are q-spaces, and regular q-spaces are quasi-k-spaces. The following lemma exhibits another class of q-spaces.

Lemma 6.1. *Every sieve-complete space Y is a q-space.*

Proof. Let $(\{U_\alpha\colon \alpha \in A_n\}, \pi_n)$ be a complete, open sieve on Y. Inductively choose $\alpha_n \in A_n$ so that $y \in U_{\alpha_n}$ and $\pi_{n+1}(\alpha_{n+1}) = \alpha_n$ for all n. Then (U_{α_n}) is complete and thus certainly countably complete. □

Remark. For regular spaces Y, Lemma 6.1 can be strengthened to conclude that Y is a space of countable type (see [1, Proposition 4.4]), and is thus, in particular, a k-space.

Theorem 6.2. *Let $f : X \to Y$ be a closed map from a sieve-complete, paracompact space X onto Y. Then* (a) *implies* (b), *and* (b)–(e) *are equivalent. If X is metrizable, then* (a)–(e) *are all equivalent*:
- (a) *Y is completely metrizable*;
- (b) *Y is sieve-complete*;
- (c) *Y is a q-space*;
- (d) Bdry $f^{-1}(y)$ *is compact for all $y \in Y$*;
- (e) *f is inductively perfect.*

Proof. (a) $\Rightarrow$ (b) Clear.

(b) $\Rightarrow$ (c) From Lemma 6.1.

(c) $\Rightarrow$ (d) By [9, Corollary 2.2], this holds for any paracompact space X. (Note that this is the only implication in this proof which depends on X being paracompact.)

(d) $\Rightarrow$ (e) Let $X' = \bigcup_y C_y$, where $C_y = \mathrm{Bdry}\, f^{-1}(y)$ if $\mathrm{Bdry}\, f^{-1}(y) \neq \emptyset$ and $C_y = \{x_y\}$ for some $x_y \in f^{-1}(y)$ otherwise. Then X' is closed in X, and $f|X'$ is a perfect map onto Y.

(e) $\Rightarrow$ (b) By [10, Theorems 6.3 and 6.5(b)].

(d) $\Rightarrow$ (a) if X is metrizable: Since X is metrizable, so is Y by the Morita–Hanai–Stone theorem [16,17] (see Engelking [5, 4.4.17]), and hence Y is completely metrizable by Theorem 1.1. □

7. Three examples

Example 7.1 below implies that the k-space assumption in Theorem 1.3 cannot be omitted, and Example 7.2 shows that, assuming $\mathfrak{b} = \mathfrak{c}$, the paracompactness assumption in Theorem 1.3 also cannot be omitted. In addition, Example 7.1 shows that the implication (a) $\Rightarrow$ (b) in Theorem 5.5 becomes false if (a) is changed to "f is irreducible". Example 7.3, finally, shows that—in contrast to Proposition 5.6 (a) $\Rightarrow$ (b)—almost complete spaces are not preserved by inductively irreducible, closed maps, even between countable metric spaces.

Example 7.1. A closed, irreducible map $f : X \to Y$ from a scattered [9] (hence partition-complete by [11, p. 514]) regular Lindelöf space X onto a space Y which is not partition-complete [10].

Proof. An example of Gruenhage and Watson [7, Example 1.2] constructs a closed map $g : A \to B$ from a scattered, regular Lindelöf space A onto a space B which is not a Baire space and hence (by [13, Propositions 4.4 and 4.5]) not partition-complete. Applying Proposition 4.4 to this $g : A \to B$ yields a map $f : X \to Y$ with the required properties. □

Example 7.2 ($\mathfrak{b} = \mathfrak{c}$). A closed, irreducible map $f : X \to Y$ from a locally compact Hausdorff (hence Čech-complete) space X onto a space Y which is not partition-complete [11].

Proof. An example of van Douwen [3, Example 13.4], which assumes $\mathfrak{b} = \mathfrak{c}$, constructs a closed map $g : A \to B$ from a locally compact Hausdorff space X onto the rationals. Applying Proposition 4.4 to this $g : A \to B$ yields a map $f : X \to Y$ with the required properties. □

Remark. The maps $f : X \to Y$ in Examples 7.1 and 7.3 are both irreducible. However, the maps $g : A \to B$ in the *proofs* of these examples are *not* irreducible, or even inductively irreducible, because every irreducible closed image (more generally: every feebly open image) of a Baire space is a Baire space; see [6, Theorem 1].

In the following example, $\mathbb{Q}$ denotes the rationals.

Example 7.3. A closed retraction (which is surely inductively irreducible) from an almost complete subspace X of $\mathbb{Q} \times \mathbb{Q}$ onto $\mathbb{Q} \times \{0\}$.

Proof. Let X be a subset of $\mathbb{Q} \times \mathbb{Q}$ which contains $\mathbb{Q} \times \{0\}$ and has a dense, discrete (hence completely metrizable) subset. Then X is almost complete. Also $\dim X = 0$ and $\mathbb{Q} \times \{0\}$ is closed in X, so there exists a closed retraction from X onto $\mathbb{Q} \times \{0\}$ by a result of Engelking [4, Lemma]. □

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[9] A space X is *scattered* if every nonempty $A \subset X$ has an isolated point.
[10] In fact, Y has a closed subspace which is not a Baire space.
[11] In fact, Y has a closed subspace homeomorphic to the rationals.

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